2. Kaleidoscopic Tilings

2.1 Tilings on surfaces

A *surface* S is a two-dimensional object such as a sphere or torus that locally looks like the plane. Every surface may be represented topologically by a sphere with attached handles. The number of attached handles is called genus of the surface, which we denote by σ. A sphere has genus 0, the torus genus 1 (see Figures 2.2 and 2.3), a surface of genus 2 is shown in Figure 2.1 below. To illustrate the origin of the word handle, you may check that the surface of a coffee cup is a surface of genus 1.

A *tiling* T of a surface S is a non-overlapping covering of the surface by polygons or *tiles.* A polygon is a (simply connected) region bounded by a finite number of curves called *edges.* The edges meet in *vertices* at definite non-zero angles. We assume that the polygon does not meet itself at a vertex or along an edge. We further assume that two intersecting tiles will either meet in a vertex or along the entire common edge of the two tiles, as illustrated in Figures 2.2, 2.3, and 2.4. Mathematically, the most interesting tilings are those that have a high degree of symmetry. We will look at *geodesic, kaleidoscopic tilings:*

- **Kaleidoscopic Condition.** A tiling is kaleidoscopic if each edge e of the tiling is a part of a closed curve, called a *geodesic* ("straight" curve), on the surface such that there is *mirror reflection* r_e of the surface in the geodesic that maps tiles to tiles. Thus the tile patterns are mirror images of each other along an edge, and in particular the angles at a vertex are all the same. The existence of the reflections also implies that the surface possesses a geometric structure – specification of geodesics, distances, angles and area – such that the geometric structure is preserved under the reflection. Except in the case of the sphere this cannot always be realized in three space with standard distance, angle and area measure and standard reflections in planes.

- **Geodesic Condition.** For each edge e of the tiling the set of fixed points of r_e, $\{x \in S : r_e(x) = x\}$ - called the mirror of the reflection - is the union of edges of the tiling. In particular, each edge is a part of a closed, smooth curve which is a union of edges.

We may understand these definitions by considering the icosahedral tiling on the sphere (Figure 2.2). Each edge of a tile is a portion of a great circle (a geodesic) obtained by intersecting the sphere with a plane passing through the center of the sphere. Reflection in this plane produces the required mirror reflection preserving angles and distances. Each great circle determined by an edge is the union of edges. The tiling of the plane by hexagons is an example of a tiling satisfying the kaleidoscopic property but not the geodesic property.

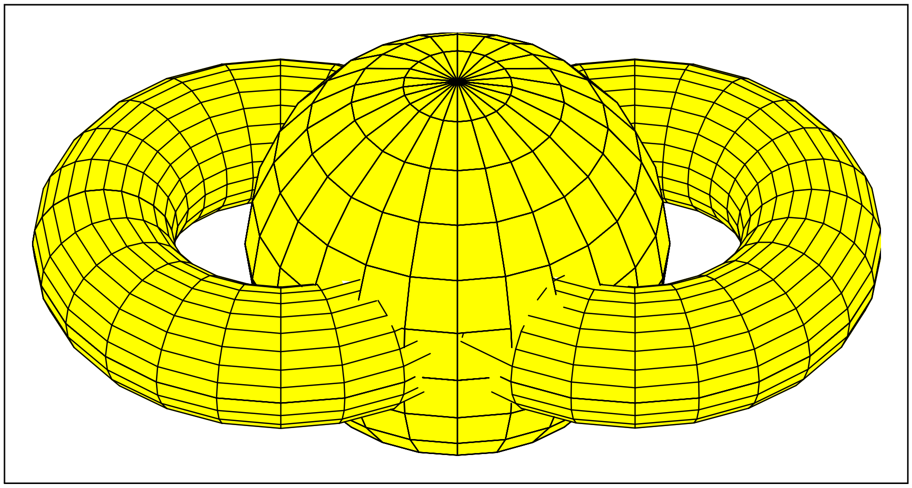

Figure 2.1 A genus 2 surface - sphere with two handles

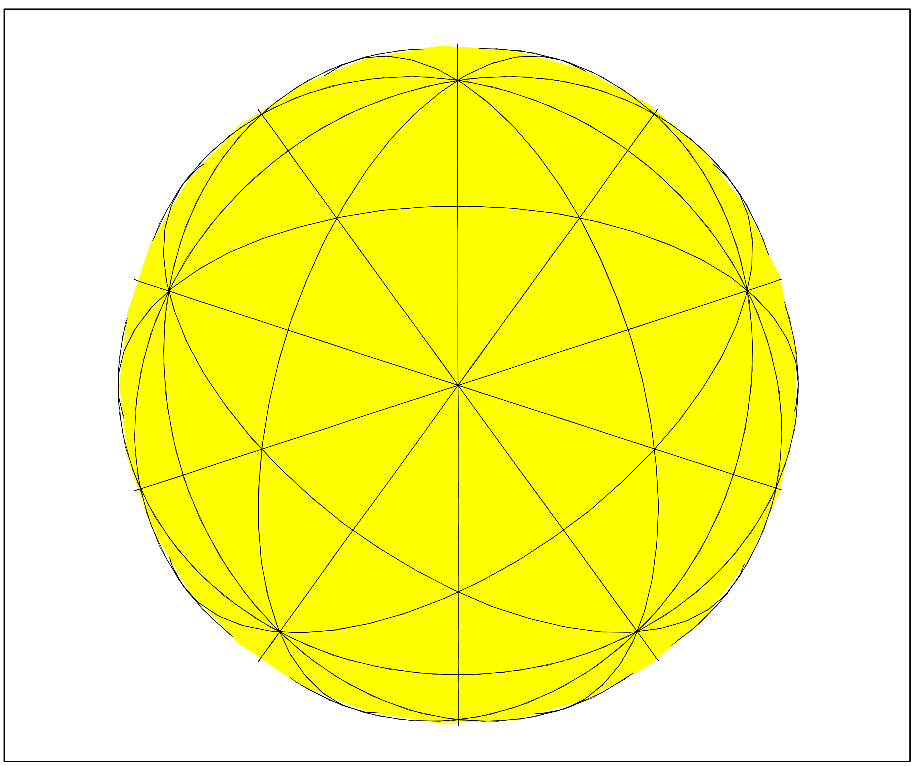

Figure 2.2 Icosahedral tiling - top view

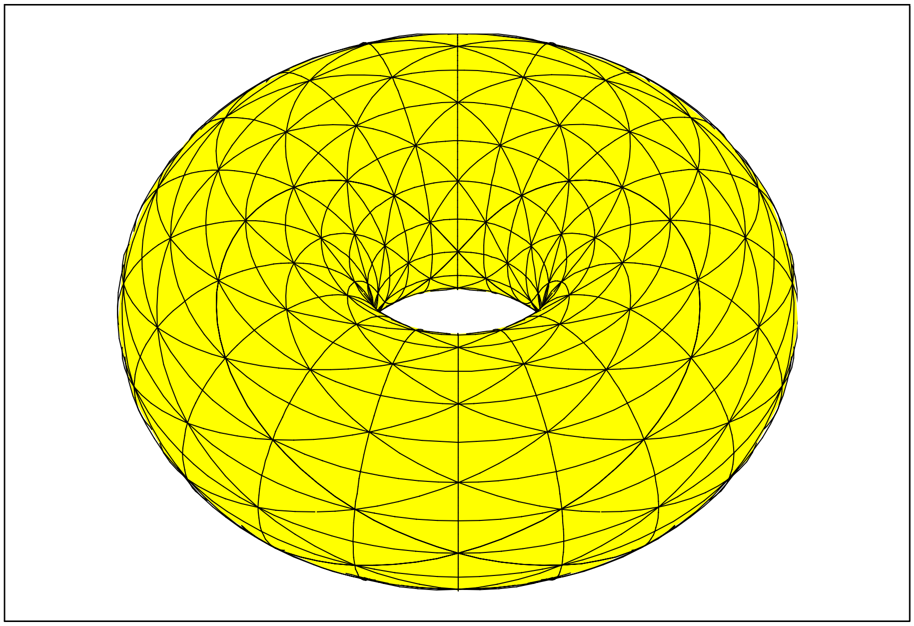

Figure 2.3 2-4-4 tiling on a torus

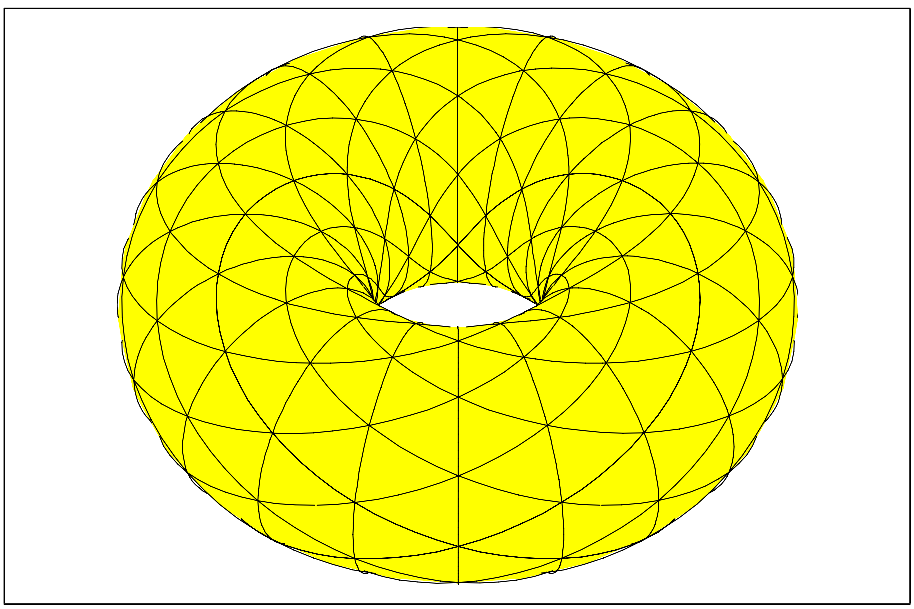

Figure 2.4 3-3-3 tiling on a torus

Remark 2.1 Tilings in which any two tiles meeting along an edge are mirror images of each other will be called locally kaleidoscopic. They are discussed in more detail in Chapter 6.

Restriction to triangles and quadrilaterals Though most the of the discussion in these notes easily extends to other polygons, we get a simplified but still very rich set of examples by focusing on tilings by simple polygons such as triangles and quadrilaterals. Though we will consider tilings by quadrilaterals later, we will keep the development simple by further restricting the initial discussion to triangles only.

2.2 Geometric considerations

The surface has a geometric structure – geodesics, distances, angles and area. For the sphere the geodesics are great circles, distance, angles and area are measured by standard calculus methods along the sphere surface, and the reflections have already been described. The specification of the geometry is more complex in the case of surfaces of positive genus and requires the notion of universal covering space, which we discuss in a later section. Now consider two edges e_1 and e_2 of a tile meeting at the vertex v, e.g., two edges meeting at the top of the sphere in the icosahedral tiling. Reflection in e_2 produces an adjacent polygon with edges $e_2' = e_2$ and e_1', meeting at v. Reflecting in e_1' provides an second polygon meeting at v, and so on. All the polygons have the same angle measure at v so it is $\frac{2\pi}{k}$, where k is the number of polygons meeting at v. By our geodesic condition the number of polygons is even, i.e., $k = 2l$ so that angle measure is $\frac{\pi}{l}$. Therefore a triangle will have angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ and angle sum $\frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n}$ for some integers l, m, n. We call such a triangle an (l, m, n)-triangle. The angle sum depends on the genus of the surface as follows.

Angle Sums

genus	geometry type	angle sum	$\frac{1}{l} + \frac{1}{m} + \frac{1}{n}$
0	spherical	$> \pi$	> 1
1	euclidean	π	1
≥ 2	hyperbolic	$< \pi$	< 1

We have already seen spherical triangles in the icosahedral tiling and euclidean triangles are very familiar to us. Some example of hyperbolic triangles are given in Figures 3.2, 3.3, 3.4, and 3.5. In our plane models of the triangles, the edges are always arcs of circles. Note that the number of possible cases for spheres and tori is quite limited, whereas for hyperbolic surfaces there are infinitely many possibilities. In our examples, the triangle generating the icosahedral tiling of the sphere is a $(2, 3, 5)$-triangle and the torus tilings are generated by $(2, 4, 4)$ and $(3, 3, 3)$ triangles. The other spherical possibilities are $(2, 2, n)$, $(2, 3, 3)$ and $(2, 3, 4)$ and the remaining torus possibility is $(2, 3, 6)$.

Each reflection has a *mirror* i.e., a set of geodesics on the surface across which the reflection occurs. The mirror is the set of points which are not moved by the reflection. For spherical tilings, the mirrors are obviously great circles. For the torus things are a bit trickier. Some of the reflections we can visualize as reflections in a plane. These mirrors have two components. With more effort we can determine the number of curves in all of the mirrors. From the examples we see that each mirror is a disjoint set of non-intersecting, smooth, closed curves each diffeomorphic to a circle. We will call the circles *ovals*. It is known how to calculate the number of ovals in a mirror, see section 8.

The geometry of tilings and the universal cover are discussed further in section 6.

2.3 The tiling group of S

Now comes the group theory. Each edge of the tiling T determines a reflection, i.e., a transformation of the surface S into itself. This transformation of the surface is an isometry, i.e., preserves distance, angle and area, and it also maps tiles to tiles. We will use these reflections to construct a group of symmetries G^* of the tiling, and use the reflection in the edges of a tile to give a special presentation of G^*. To this end, select a tile Δ_0, which we call the master tile, as shown in Figure 2.5 below. The triangle has been drawn with curved sides to suggest a hyperbolic triangle on a surface of genus ≥ 2. We denote the sides by p, q, r, and we also let p, q, r denote the three corresponding reflections in the sides Δ_0 – see Figure 2.5. The reflected images, $p\Delta_0$, $q\Delta_0$, and $r\Delta_0$, of Δ_0 in the sides p, q, and r have been drawn with dotted lines. As Δ_0 is an (l, m, n)-triangle then the product $a = pq$ is easily seen to be a counter clockwise non-euclidean rotation through $\frac{2\pi}{l}$ radians, mapping $q\Delta_0$ onto $p\Delta_0$. Similarly, for $b = qr$ and $c = rp$ are counterclockwise rotations through $\frac{2\pi}{m}$ radians and $\frac{2\pi}{n}$ radians respectively.

From these observations and the fact that reflections have order 2, we get the following:

$$a^l = b^m = c^n = 1, \tag{1}$$

and

$$abc = 1, \tag{2}$$

since $pqqrrp = 1$.

Let $G^* = \langle p, q, r\rangle$ and $G = \langle a, b, c\rangle = \langle a, b\rangle$ be the groups generated by the above elements. The subgroup G is the subgroup of conformal (orientation-preserving) isometries in G^*, G is normal in G^* of index 2, in fact $G^* = \langle q\rangle \ltimes G$, a semi-direct product.

Definition 2.1 The group G^* defined above will be called the tiling group of S (with its given tiling). The subgroup G will be called the *conformal tiling group.*

Remark 2.2 There may be isometries of S that preserve the tiling but are not contained in G^*. In this case the total symmetry group of the tiling has a factorization UG^*, where U is the stabilizer of the master tile. Since U is a group of isometries of a tile it is a subgroup of Σ_3, the symmetric group on the vertices of the master tile. Since the term “symmetry group” of an object usually refers to the totality of self-transformations of the object preserving the given structure the use of the term symmetry group to describe G^* would be misleading. Therefore, we use the term tiling group.

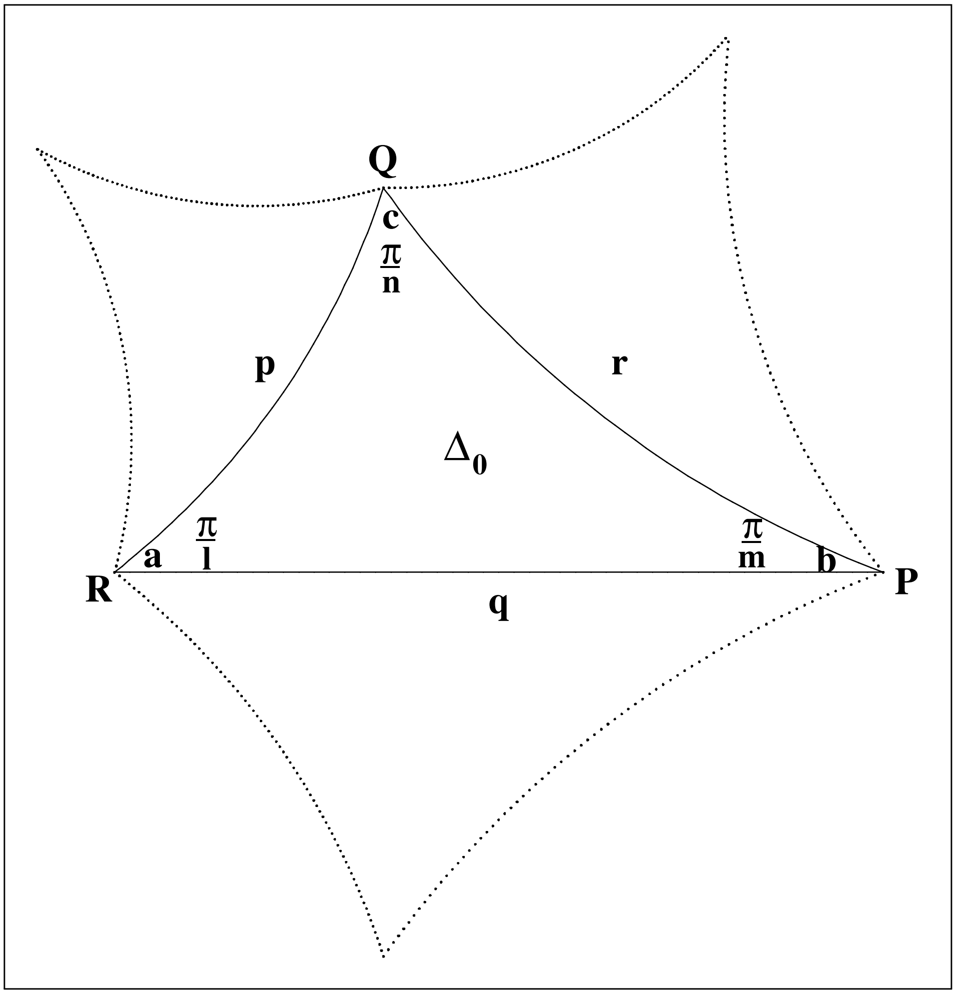

Figure 2.5 The master tile, reflected images, and group generators

The conjugation action of q on the generators a, b of G induces an automorphism θ satisfying:

$$\theta(a) = qaq = qaq^{-1} = a^{-1}, \tag{2.3}$$

$$\theta(b) = qbq = qbq^{-1} = b^{-1} \tag{2.4}$$

The relation between the group order G and the genus σ of the surface is given by the Riemann-Hurwitz equation:

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right). \tag{5}$$

It follows that the genus is given by:

$$\sigma = 1 + \frac{|G|}{2}\left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right), \tag{6}$$

and the group order by:

$$|G| = \frac{2\sigma - 2}{1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)}. \tag{7}$$

A table of values of some values of $\mu(l,m,n) = \frac{2\sigma-2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)$ is in the appendix at the end of the notes.

The triple of elements (a,b,c) of elements from G which generates G and satisfies (2.1) and (2.2) is a called a *generating* (l,m,n) *-triple* of G. Just as we may create a triple from a tiling, a tiling may be created from triples as in the following theorem.

Theorem 2.1 *Let G have a generating (l,m,n)-triple and suppose that the quantity σ defined by (2.6) is an integer. Then there is always a surfaces S of genus σ with an orientation preserving G-action. If, in addition, there is an involutary ($\theta^2 = id$) automorphism θ of G satisfying (2.3) and (2.4), then the surface S has a tiling T by (l,m,n)-triangles such that orientation preserving tiling group as constructed above is the original G, and such that $G^* \simeq \langle\theta\rangle \ltimes G$.*

Remark 2.3 The elements of $\langle\theta\rangle \ltimes G$ are of the form g or θh, $g,h \in G$. The only non-obvious multiplications are $\theta g \theta h = \theta g \theta^{-1} h = \theta(g)h$ and $g\theta h = \theta\theta g \theta h = \theta\theta(g)h$. The formulae giving p,q,r in terms of θ and a,b,c are:

$$\begin{aligned} p &= \theta a^{-1} \\ q &= \theta \\ r &= \theta b \end{aligned} \tag{2.8}$$

We say that the tiling of S is *induced by the triple* (a,b,c).

Remark 2.4 It can be shown that if the generating triple exists, then σ is automatically an integer.

Remark 2.5 If the automorphism θ does not exist then S still carries an (l,m,n)-tiling. However the tiling is only locally kaleidoscopic not globally kaleidoscopic. The group G is still generated by rotations at the corners of a tile, but none of the local reflections extend to a global reflection. The ovals generated by an edge still exist, but the mirror generated by the refection in an edge is not defined,

Example 2.1 Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. The group G has many generating $(4,4,4)$-triples (96 as a matter of fact) and the inversion automorphism $\theta : (x,y) \to (-x,-y)$ satisfies the properties required by (2.3) and (2.4). Thus, there is at least one tiling of a surface of genus 3 by 32, $(4,4,4)$-triangles.

The above example prompts the following question. How many of the 96 different triples yield different tilings. This is answered by the following definition and theorem.

Definition 2.2 Suppose that G^* acts on two surfaces S, S' defined by tilings on the surfaces. Then we say that the actions are *isometrically equivalent* if there is an isometry $h : S \to S'$ and an automorphism ω of G^* such that

$$h(g \cdot x) = \omega(g) \cdot h(x) \text{ for all } x \in S. \tag{9}$$

If in addition S and S' have assigned orientations and h is orientation preserving then h is said to be a *conformal equivalence* and that the actions of G^* are *conformally equivalent*. Similar definitions hold for the action of G.

Theorem 2.2 *Let (a, b, c) be a generating (l, m, n)-triple of G and let θ be an involutary automorphism of G satisfying (2.3) and (2.4). Let ω be an automorphism G and let $a' = \omega(a), b' = \omega(b), c' = \omega(c)$, and $\theta' = \omega\theta\omega^{-1}$. Then for any two surfaces S and S' with (l, m, n)-tilings induced by the triples (a, b, c) and (a', b', c'), respectively, there is an isometric equivalence $h : S \to S'$ satisfying (2.9) above. (If the genus of S is 1, then S and S' must also have the same area.)*

Remark 2.6 An isometry which is not conformal preserves the measure of angles but reverses their sense. Therefore, it is called anti-conformal. Actions of G^* that are isometrically equivalent are automatically conformally equivalent since we may replace h and ω by the $h' = hp$ and ω by $\mathrm{Ad}_p \circ \omega$, where $\mathrm{Ad}_p(g) = pgp^{-1}$. On the other hand, actions of G that are isometrically equivalent are not necessarily conformally equivalent. Finally, an isometric equivalence of G-actions induces an equivalence of G^*-actions provided that one of the G-actions extends to a G^*-action.

The tiles of a surface may be uniquely labelled by the elements of G^*. Theorem 2.2 says that any relabelling the tiles by an automorphism of G^* is actually induced by an isometry of the surface. A complete statement is the following.

Theorem 2.3 *Let (a, b, c) be an (l, m, n)-generating triple for G corresponding to the surface S. Then the (l, m, n)-generating triple (a', b', c') corresponds to a conformally equivalent surface, S', with conformal equivalence $h : S \to S'$ commuting with the group actions of G, if and only if one of the following holds:*

- $l < m < n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that $(a', b', c') = \omega \cdot (a, b, c)$,
- $l = m < n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of

$$\begin{aligned}(a', b', c') &= \omega \cdot (a, b, c),\\ (a', b', c') &= \omega \cdot (aba^{-1}, a, c)\end{aligned}$$

  holds,
- $l < m = n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of

$$\begin{aligned}(a', b', c') &= \omega \cdot (a, b, c),\\ (a', b', c') &= \omega \cdot (a, bcb^{-1}, b)\end{aligned}$$

  holds,

- $l = m = n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of

$$\begin{aligned}(a', b', c') &= \omega \cdot (a, b, c),\\ (a', b', c') &= \omega \cdot (aba^{-1}, a, c),\end{aligned}$$

$$\begin{aligned}(a', b', c') &= \omega \cdot (a, bcb^{-1}, b),\\ (a', b', c') &= \omega \cdot (c, b, cac^{-1}),\end{aligned}$$

$$\begin{aligned}(a', b', c') &= \omega \cdot (b, c, a),\\ (a', b', c') &= \omega \cdot (c, a, b)\end{aligned}$$

  holds.

For anti-conformal equivalences we have a similar theorem.

Theorem 2.4 *Let notation be as in Theorem 2.3 above then there is an anti-conformal equivalence $h : S \to S'$ commuting with the group actions of G if and only if.*

- $l = m < n$ and there is an $\omega \in Aut(G)$ such that

$$(a', b', c') = \omega \cdot (b^{-1}, a^{-1}, c^{-1})$$

  holds,

- $l < m = n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that

$$(a', b', c') = \omega \cdot (a^{-1}, c^{-1}, b^{-1})$$

  holds,

- $l = m = n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of

$$\begin{aligned}(a', b', c') &= \omega \cdot (b^{-1}, a^{-1}, c^{-1}),\\ (a', b', c') &= \omega \cdot (a^{-1}, c^{-1}, b^{-1}),\\ (a', b', c') &= \omega \cdot (c^{-1}, b^{-1}, a^{-1})\end{aligned}$$

  holds.

The key point - the geometry and group theory connection. The above construction and theorems allows us to describe a tiling, and its geometric properties by the group G, the generating triple (a, b, c), the automorphism θ and p, q, and r. We may also do computations in the group to discover various properties of the tiling, and to discover when two tilings are the same. Many of the warm-up problems in the next chapter develop the key point. For example, the length of ovals may be calculated by computing the order of certain elements in the group. The set of all possible G for surfaces of genus 2 and 3 are contained in the tables in [2]