5. Group Actions and Group Constructions

5.1 Group actions

Definitions and results A group G acts on a set X if there is a map $G \times X \to X$, $(g, x) \to gx$ satisfying:

$$\begin{aligned} g(hx) &= (gh)x \\ 1x &= x, \end{aligned}$$

where $1 \in G$ is the identity element. The action is said to be *effective* if $gx = x$ for all $x \in X$ implies that $g = 1$. The action is said to be *transitive* if for every $x, y \in X$ there is a $g \in G$ such that $y = gx$. The *orbit* and *stabilizer* of $x \in X$ are the set $Gx \subseteq X$ and subgroup $G_x \subseteq G$, respectively, defined by:

$$\begin{aligned} Gx &= \{gx : g \in G\}, \\ G_x &= \{g \in G : gx = x\} \end{aligned}$$

and more generally for $Y \subseteq X$

$$G_Y = \{g \in G : gx = x, \forall x \in Y\}$$

Analogously, for $H \subseteq G$ and $g \in G$ the fixed point subsets X^H and X^g are defined by:

$$\begin{aligned} X^H &= \{x \in X : gx = x, \forall g \in H\}, \\ X^g &= \{x \in X : gx = x\}. \end{aligned}$$

The action is transitive if X is a single orbit. The following formula is useful and is easily established:

$$G_{gx} = gG_xg^{-1}. \tag{1}$$

The *orbit-stabilizer theorem* is the most useful result about group actions. The theorem states the map $gG_x \to gx$ is a $1-1$ correspondence between the coset space $G/G_x = \{gG_x, g \in G\}$ and the orbit Gx. It follows that

$$|Gx| = [G : G_x],$$

or if G is finite then

$$|Gx| = \frac{|G|}{|G_x|}.$$

Let $\sum_X$ denote the group of permutations of X, i.e., 1-1 mapping of X onto itself. Then there a homomorphism map $\pi : G \to \sum_X$, $g \to \pi_g$ given by $\pi_g(x)$ =gx. The kernel K of this homomorphism is the subgroup K defined by

$$K = \{g \in G : gx = x, \forall x \in X\}.$$

If the action is effective the K is trivial and we say that π is a faithful permutation representation. If G acts transitively, i.e., $X = Gx$ then

$$K = \bigcap_{g \in G} g G_x g^{-1} = G_X.$$

Finally the number of orbits may be determined by the Polya-Burnside theorem. It says that the number of G-orbits in X is a fixed point sum.

$$\frac{1}{|G|} \sum_{g \in G} |X^g| .$$

This is easily proved for transitive actions from the orbit stabilizer theorem by calculating the cardinality of the set $Z = \{(g, x) \in G \times X : gx = x\}$ in two ways.

$$\sum_{g \in G} |X^g| = |Z| = \sum_{x \in X} |G_x| .$$

The general result follows from applying the special case to each orbit.

Examples of actions Of course the results on actions become more useful when applied to concrete examples. Here is a list of examples of actions.

- $G \subseteq \sum_n$, $X = \{1, \ldots, n\}$
- G is any finite group and $X = G$. Then, G acts on itself by conjugation and the orbits are the conjugacy classes. The Polya -Burnside theorem says that the number of conjugacy classes in a group G is:

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Cent}_G(g)| .$$

- G is any finite group and X is any set of subgroups of G that is invariant under conjugation, e.g., cyclic groups of the same order, p-groups of the same order.
- G is a subgroup of the "$Ax + b$" group on a vector space $V = k^n$ over a finite field $k = \mathbb{F}_q$. Let b be a vector in V and A an $n \times n$ matrix with entries in k. The the group element (A, b) acts on $x \in V$ by $x \to Ax + b$. Computation shows that the multiplication is the semi-direct product multiplication.

$$(A, b) \cdot (C, d) = (AC, Ad + b).$$

  Now we may pick X to be the points of V, the lines of V or the s -dimensional affine subspaces of V.
- Projectivize the above affine linear example.
- Same as the proceeding examples but now act on triangles, conic sections and other geometric objects. Note the triangle action is the action of G on $V \times V \times V$.

- G is a tiling group on a surface S and X is the set of tiles, edges or vertices.

Functorial constructions

- G acts on X and H acts on Y. Then $G \times H$ acts on $X \times Y$ by $(g,h)\cdot(x,y) = (gx, hy)$.
- G acts on X and Y. Then G acts $X \times Y$ by $g \cdot (x,y) = (gx, gy)$.
- G acts on X. Then G acts on X^n by $g \cdot (x_1, \ldots, x_s) = (gx_1, \ldots, gx_s)$.
- Same as above except we only look at s-tuples without repeats.
- G acts on X. Then G acts on the s-subsets of X by $g \cdot \{x_1, \ldots, x_s\} = \{gx_1, \ldots, gx_s\}$

5.2 Semi-direct products

Internal semi-direct products We develop the idea of semi-direct products by generalizing direct products. The examples below may be illustrate the abstract idea. Suppose that G is a group with two subgroups H, N such that, $G = HN$, $H \cap N = \{1\}$ and the elements of H commute with those of N. Then the map $H \times N \to G$, $(h,n) \to hn$ is an isomorphism since $h_1n_1h_2n_2 = h_1h_2n_1n_2$. We also say that G is an internal direct product of H and N. Observe that the maps $G \to H$, and $G \to N$ given by $g = hn \to h$ and $g = hn \to n$ are surjective homomorphism with kernels N and H respectively, so that $G/N \backsimeq H$ and $G/H \backsimeq N$. We say that H is a complementary subgroup to N and vice versa.

Remark 5.1 Note that if we relax commutativity condition to the property that N and H normalize each other we still get that G is an internal direct product of H and N.

Now suppose that we have the following situation $N \lhd G$. Does there exist a subgroup H such that $G/N \cong H$, i.e., a complementary subgroup? The answer is not always, and even if there is, G is not necessarily an internal direct product. Thus let us relax our conditions to the following:

$$N \lhd G, \; G = HN, \; H \cap N = \{1\},$$

and see how far we can get. Note that every element in G has a factorization $g = hn$, $h \in H$, $n \in N$, because $G = HN$. The factorization is unique since two different factorizations $g = h_1n_1 = h_2n_2$, yields the element $h_2^{-1}h_1 = h_2^{-1}gn_1^{-1} = n_2n_1^{-1} \in H \cap N = \{1\}$. Thus there is a set map $H \times N \to G$, $(h,n) \to hn$. It is not a homomorphism if given by$H \times N$ is given the product group structure, but it will be if we give $H \times N$ the multiplication

$$(h_1, n_1) \cdot (h_2, n_2) = (h_1h_2, h_2^{-1}n_1h_2n_2)$$

as suggested by the valid formula in G.

$$h_1n_1h_2n_2 = h_1h_2h_2^{-1}n_1h_2n_2.$$

It is not hard to show that the multiplication above is associative that $(1_H, 1_N)$ is the identity element an that the inverse is given by

$$(h,n)^{-1} = (h^{-1}, hn^{-1}h^{-1})$$

and that the multiplication map is an isomorphism. We say that G is an internal semi-direct product of H and N and we write $G = H \ltimes N$. Now let us start with an arbitrary H and N and try to construct a G. Before doing this let us write $n^h = h^{-1}nh$ to denote the conjugation action of h^{-1} on n and note some "power" properties of n^h. Let $h, k \in H$ and $m, n \in N$. Then,

$$\begin{aligned} n^h &\in N, \\ (mn)^h &= m^h n^h, \\ n^{hk} &= (n^h)^k. \end{aligned} \tag{5.2}$$

Now let $\Theta : H \to \mathrm{Aut}(N)$, $h \to \Theta_h$ be any homomorphism. Define n^h in this situation by:

$$n^h = \Theta_{h^{-1}}(n).$$

Then the formulas in 5.2 still hold. In fact the map $H \to \mathrm{Aut}(N)$ in the internal case is $h \to \mathrm{Ad}_h$ where $\mathrm{Ad}_h(n) = hnh^{-1}$. Using the formulas in 5.2 it is not hard to show that we may define a group structure on the set $H \times N$ by

$$(h_1, n_1) \cdot (h_2, n_2) = (h_1h_2, n_1^{h_2}n_2) \tag{3}$$

We denote the set $H \times N$ with this multiplication by $H \ltimes N$ and call it the external semi-direct product of H and N.

Remark 5.2 The semi-direct product of H and N is not unique since there may be many eligible homomorphisms $\Theta : H \to \mathrm{Aut}(N)$. Note that the trivial action $n^h = h$, is always a possibility for Θ and that it yields the direct product.

Remark 5.3 By writing elements in the form nh we arrive at an alternative multiplication rule on $N \times H$ with formula

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 n_2^{h_1^{-1}}, h_1h_2) = (n_1h_1n_2h_1^{-1}, h_1h_2).$$

In this case we write $N \rtimes H$ for the semi-direct product.

Examples of semi-direct products.

Example 5.1 Let G be the group of symmetries of a regular n-gon. Ley y be a rotation through $\frac{2\pi}{n}$ and let r be reflection in the x-axis. Then it si easy to show that $ryr^{-1} = y^{-1}$. Since $\langle r \rangle \cap \langle y \rangle = \{1\}$ then $G = \langle r \rangle \ltimes \langle y \rangle$.

Example 5.2 Let k be any field and let $V = k^n$. Let N denote the additive group of V and let H be a group of $n \times n$ matrices. The group H acts as a group of automorphisms of N by $b^A = Ab$ for $A \in H$, $b \in N$. Now the semi-direct product $N \rtimes H$ may be viewed as a group of transformations of V by: $(b, A) \cdot x = Ax + b$ for $x \in V$. The multiplication formula given by the action comes from

$$((b, A) \cdot (d, C)) \cdot x = A(Cx + d) + b = (b + Ad, AC) \cdot x.$$

Example 5.3 Let $H = \mathbb{Z}_p = \langle x \rangle$, and $N = \mathbb{Z}_q = \langle y \rangle$ be cyclic groups of order p and q respectively - p and q need not be primes. Let r be an integer satisfying $r^p \equiv 1 \mod q$. It automatically follows that r and q are relatively prime and hence r represents an element of $\mathbb{Z}_q^*$ of order dividing p. Thus action of H on N may be defined by $y^x = y^r$. The semi-direct product then has the presentation

$$D_{p,q,r} = H \ltimes N = \langle x, y : x^p = y^q = 1, y^x = y^r \rangle .$$

Such groups are called split metacyclic groups. They are extremely useful in constructing examples and counterexamples since they are not abelian, but are easy to compute with. The dihedral groups above are specific examples of this type of semi-direct product.

Example 5.4 Let p be a prime $N = \mathbb{Z}_p, H = \mathbb{Z}_p^*$. Then $H \ltimes N$ may be represented as an $Ax + b$ group with $n = 1$ or as split metacyclic group as above

5.3 Permutation Groups

Since we frequently represent our groups as permutation groups it is useful to use the permutation representation to deduce some group structure. To this end let us suppose that G is a group with a faithful, transitive permutation representation on a set X. Let $H = G_{x_0}$ be the stabilizer of a given point x_0. Since G acts faithfully and transitively then

$$\bigcap_{g \in G} gHg^{-1} = \{1\}.$$

A invariant block structure on X is a non-trivial partition P of X into at disjoint subsets $X = \dot{\bigcup_i} X_i$, called blocks, that are permuted by G. The blocks can neither be singletons nor all of X. If X admits no such partition then G is said to be primitive of degree $|X|$. A lot is known about primitive groups of low degree so that we may be able to determine G.

Let X_0 be the block containing x_0. Let $M = \{g \in G : gX_0 = X_0\}$. Then $H \subsetneqq M \subsetneqq G$ and $X_0 = Mx_0$, and the block are in $1-1$ correspondence to the cosets of M. Clearly a subgroup M lying strictly between H and G will also define such a block structure. Now consider the permutation representation $\pi : G \to \sum_P$ afforded by the blocks. The kernel K of this map is:

$$K = \bigcap_{g \in G} gMg^{-1},$$

which conceivably could be trivial. In any event we have an exact sequence: $K \hookrightarrow G \twoheadrightarrow G/K$ which might give us some structure on G. We are in particularly good shape if $H \lhd M$. One way for this to happen is if $M = N_G(H)$ when H is not self-normalizing. Now let us assume that $H \lhd M$ and let $N = M/H$. We can define a embedding from K into $N^{|P|}$ as follows. Let $g_1, \ldots, g_p$, $p = |P|$ be a set of coset representatives for M in G. Now for $g \in K$, $g \in g_i M g_i^{-1}$ for each g_i. Thus $\phi_i(g) = g_i g g_i^{-1} H$ is an element of $N = M/H$. Now let $\Phi : K \to N^p$ be defined by $\Phi(g) = (\phi_1(g), \ldots, \phi_p(g))$. $\Phi(g)$ maps to zero if and only if $g_i g g_i^{-1} \in H$ for all g_i. Using the fact that $H \lhd M$, it is easily shown that

$$\bigcap_{i=1}^{p} g_i H g_i^{-1} = \bigcap_{g \in G} g H g^{-1} = \{1\}.$$

Thus it follows that Φ is a 1-1 mapping. Further note that $\Phi(K)$ an invariant G/K subgroup of $N^{|P|}$. This significantly restricts the form of $\Phi(K)$.

5.4 Matrix Groups

Let k be a field, then we can define various matrix groups:

- $GL_n(k) = \{n \times n \text{ invertible matrices with entries in } k\}$,
- $SL_n(k) = \{A \in GL_n(k) : \det(A) = 1)\}$,
- $O_n(k) = \{A \in GL_n(k) : A^{-1} = A^t\}$,
- $SO_n(k) = \{A \in SL_n(k) : A^{-1} = A^t\}$.

Note that each of these groups acts on points, lines, planes and other objects in k^n. We are particularly interested in the case where $k = \mathbb{F}_q$ the finite field of q elements.

5.5 Projective linear groups

Let k be a field and $GL_2(k)$ as defined above. Let ℓ be a non-horizontal line passing through the origin in k^2 and let $(x, y) \in \ell$. Then the quantity $z = \frac{x}{y}$ is independent of the point (x, y) and is the reciprocal of the slope. Fora vertical line let $z = \infty$. The set $\widehat{k} = k \cup \{\infty\}$ is called the projective line over k and is an analog of $\widehat{\mathbb{C}}$ that we defined in the chapter on hyperbolic geometry. If ℓ_z denotes the line with reciprocal slope z then Then $\{\ell_z : z \in \widehat{k}\}$ is the totality of lines in k^2.

The group $GL_2(k)$ acts on $\widehat{k}$ by transferring the action of $GL_2(k)$ on $\{\ell_z : z \in \widehat{k}\}$ to $\widehat{k}$ as follows. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(k)$, and let $(x, y) \in \ell_z$. Let (x', y') be the corresponding point on $\ell_{z'} = A\ell_z$. Then

$$z' = \frac{x'}{y'} = \frac{ax + by}{cx + dy} = \frac{a\frac{x}{y} + b}{c\frac{x}{y} + d} = \frac{az + b}{cz + d}.$$

This action has exactly the same as the linear fractional transformation $T_A(z)$ defined and discussed in section 4.6. It is not too hard to show the following properties. The homomorphism

$$GL_2(k) \longrightarrow \mathrm{Aut}(\widehat{k}),\ A \longrightarrow T_A.$$

has kernel exactly equal k^*I the set of invertible scalar matrices. The image group, considered as a group of transformations of $\widehat{k}$ or as the quotient group $\frac{GL_2(k)}{k^*I}$ is called the projective linear group and is denoted by $PGL_2(k)$. Since the only scalar matrices with determinant 1 are $\pm L$ then $SL_2(k) \cap k^*I = \{+I, -I\}$ and so we define $PSL_2(k)$ as $SL_2(k)/\{+I, -I\}$. This subgroup is of index 2 in $PGL_2(k)$, unless k has characteristic 2. Finally we note that $PGL_2(k)$ acts sharply triply transitively on $\widehat{k}$. This means that for any pair of triples of distinct points (z_1, z_2, z_3) and (w_1, w_2, w_3) there is a unique element of $A \in PGL_2(k)$ that maps z_i to w_i. Of course the greatest interest in $PGL_2(k)$ and $PSL_2(k)$ are when k is a finite field.