Higher genus Soccer Balls and Kaleidoscopic Tilings in the Hyperbolic Plane

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Outline

- **Talk 1** The relation between higher genus soccer balls and the kaleidoscopic tilings
- **Talk 2** Divisible tiling in the hyperbolic plane
- all of this work has been done jointly with undergraduates
First Talk: Soccer Ball and Tilings

- Analyze soccer ball (and torus soccer ball)
- Riemann-Hurwitz equation and Euler characteristic
- Tilings in the Euclidean and hyperbolic plane
- Making higher genus soccer balls
- Geometric structure vs. algebraic structure
  - Punch line: Group theory wins the day
The soccer ball

• show ball
• show tiling of soccer ball
• show tiling of torus
• define tiling
• from tiling to soccer ball
Icosahedral-Dodecahedral Tiling
(2,3,5) - tiling
(2,4,4) - tiling of the torus
Tiling: Definition

- Let $S$ be a surface of genus $\sigma$.
- **Tiling:** Covering by polygons “without gaps and overlaps”
- **Kaleidoscopic:** Symmetric via reflections in edges.
- **Geodesic:** Edges in tiles extend to geodesics in both directions
- terminology: $(l,m,n)$ -triangle,
Tiling to soccer ball

• standard soccer ball example
• non-standard soccer ball
• torus example
Riemann Hurwitz equation
(euler characteristic proof)

Let \( S \) be a surface of genus \( \sigma \) and \( 2|G| \) the number of triangles:

\[
\frac{2\sigma - 2}{|G|} = 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}
\]
Hyperbolic triangles

- hyperbolic when \( \sigma \geq 2 \) or

\[
\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 0
\]
Hyperbolic geometry

- Points, lines and angles
- Reflections - show picture
Tilings of Euclidean and Hyperbolic Plane - Examples

- Euclidean
  - (2,3,6) example
  - (2,4,4) example
  - (3,3,3) example

- hyperbolic
  - (2,3,7) example
  - (3,3,4) example
Soccer Ball Patterns

- (2,3,6) patterns
- (2,4,4) patterns
- (2,3,7) pattern
Making Soccer Balls

- Show picture of making surface from a tiling
- words to label tiles
- word relations give a surface construction recipe
Word relations 1

Define:

\[ a = pq, \ b = qr, \ c = rp \]

There are universal relations and surface relations
$p^2 = q^2 = r^2 = 1,$
$a^l = b^m = c^n = 1,$
$a b c = 1, (p q q r r p = 1)$
$\theta (a) = q a q^{-1} = q p q q = q p = a^{-1},$
$\theta (b) = q b q^{-1} = q q r q = r q = b^{-1}.$
Word relations 3
Additional relations for a surface

Example Suppose we have \( l=m=n=4 \) and \( ab=ba \) then \( |G| = 16 \).

Reason: There is a map
\[
\{\text{words in } a, b, c\} \longrightarrow G = Z_4 \times Z_4
\]
where \( Z_4 = \text{integers mod 4} \).
Punch line

- Group theory exactly explains the structure of tiled surfaces and soccer balls
- The very rich theory of groups gives a lot of power in exploring the structure of tilings and soccer balls