Higher Genus Soccer Balls

S. Allen Broughton

Rose-Hulman Institute of Technology
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Outline

Credits

- most work done with undergraduates in RHIT NSF-REU
- research site for the work http://www.tilings.org/
- also joint work with Aaron Wooton at University of Portland

why soccer balls - 1

- Here is a picture of a soccer ball borrowed from everbe.com via Google.

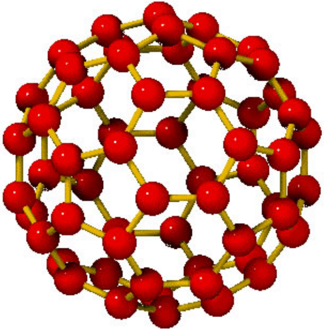
- observe that the pentagons and hexagons are regular
- there are twelve pentagons and twenty hexagons

why soccer balls - 2

- motivation from chemistry
  - soccer ball is a model for bucky balls and nanotubes
  - see pictures next two slides
- motivation from fibre arts
  - construction of Tamari balls, see later slides
  - baseball caps - questions from a cap maker

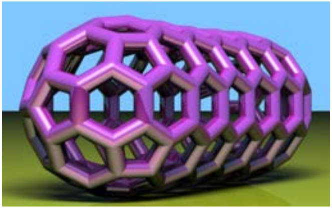
why soccer balls - 3

- Here is a ball and stick model of a bucky ball from the site http://www.psyclops.com/bucky.shtml



- there are 60 atoms and 90 bonds
- chemists observe observe that the pentagons are regular but the hexagons are not

why soccer balls - 4

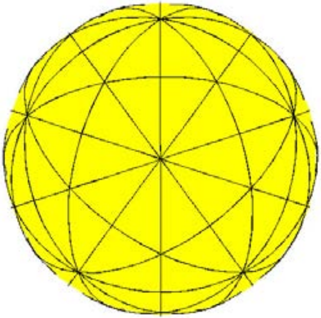
- Here is a cartoon model of a small single walled carbon nanotube (SWT) from the site http://www.icpf.cas.cz/jiri/pictures/nanotube.jpg
  
- there are 12 pentagons and some number of hexagons

why soccer balls - 5

- Here is a picture of some Tamari Balls borrowed from http://www.istockphoto.com/

kaleidoscopic tilings - picture

- consider tiling by (2,3,5)-triangles on the sphere



- reflections in the edges of triangles preserve the tiling

kaleidoscopic tiling - definition

- S is a surface of genus σ (assume the sphere for the moment)
- Tiling: covering of S by polygons “without gaps and overlaps”
- Kaleidoscopic: symmetric via reflections in edges
- Geodesic: edges in tiles extend to geodesics in both directions

kaleidoscopic tilings on the sphere

- Terminology: (ℓ, m, n)-triangle, angles are π/ℓ, π/m, π/n
- kaleidoscopic triangles on the sphere $(2, 2, d)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$
- we will see these in a moment when we visit the soccer ball page

kaleidoscopic tilings - number of triangles

- Let S be a surface of genus σ and $2|G|$ the number of triangles
- then

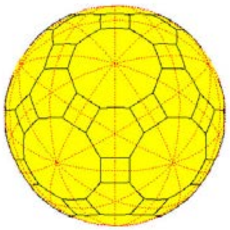
$$\frac{2\sigma - 2}{|G|} = 1 - \frac{1}{\ell} - \frac{1}{m} - \frac{1}{n}$$

Cayley Graph construction - 1

- select a distinguished tile called the master tile.
- select a point in master tile
- reflect the point in three sides of the triangle
- join the new points to the original point by line segments
- repeat until no more new points

Cayley Graph construction - 2

- example for $(2, 3, 5)$ tiling



- next visit soccer ball page http://www.rose-hulman.edu/ brought/Epubs/soccer/soccer.html
- during visit observe that torus examples are derived from tilings and Cayley graphs on the plane

Hyperbolic geometry - 1

- show big picture: t433.pdf

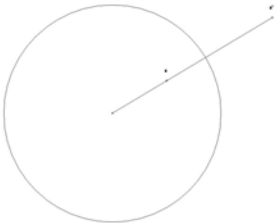
Hyperbolic geometry - 2

- from the big picture
- points: interior points of unit circle
- lines: circles perpendicular to boundary of the unit circle or a diameter
- angles: angle of intersection via calculus
- distance: ghastly formula

Hyperbolic geometry - 3

- reflections: inversion in a circle

$$|z - O||z' - O| = r^2$$



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Hyperbolic geometry - 4

- for hyperbolic triangles sum of angles less that 180 so

$$\frac{\pi}{\ell} + \frac{\pi}{m} + \frac{\pi}{n} < \pi$$

- both sides of following equation are positive when $\sigma > 2$

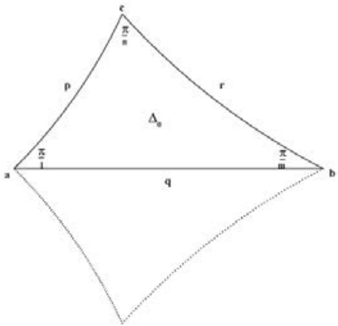
$$\frac{2\sigma - 2}{|G|} = 1 - \frac{1}{\ell} - \frac{1}{m} - \frac{1}{n}$$

Hyperbolic surfaces

- the surfaces with the tilings are not easily representable in a "geometrically faithful" way in three dimensions
- as in the case of the torus represent as a finite collection of tiles suitably rolled up
- see the further questions section

tiling group construction - 1

- G^* group generated by reflections in edges of a tile - called a reflection group
- G is the subgroup of group G^* which are orientation preserving - called a rotation group
- describe group by means of the following picture



tiling group construction - 2

- define

$$a = pq, b = qr, c = rp$$

-

$$G^* = \langle p, q, r \rangle$$
$$G = \langle a, b, c \rangle$$

-

$$p^2 = q^2 = r^2 = 1$$
$$a^\ell = b^m = c^n = abc = 1$$

-

$$\theta(a) = qaq = qpqq = qp = a^{-1}$$
$$\theta(b) = qbq = qqrq = rp = b^{-1}$$

tiling groups

- find
$$G = \langle a, b, c | a^\ell = b^m = c^n = abc = 1 \rangle$$
- and an automorphism θ
-
$$\theta(a) = qaq = qpqq = qp = a^{-1}$$
$$\theta(b) = qbq = qqrq = rp = b^{-1}$$
- always true for abelian groups
- frequently true for general groups

tiling groups - some results - 1

- all kaleidoscopic triangular tilings up to genus 25 have been determined
- focus of REU program - how I met Robert J.

tiling groups - some results - 2

- an investigation by quadrilaterals has been started
- all kaleidoscopic quadrangular tilings up to genus 13 have been determined (REU program)
- with Aaron Wooton some general results on abelian groups have been obtained
- all kaleidoscopic quadrilateral tilings with G = elementary abelian group have been determined.

Further questions - classification

- complete classification of all tilings for low genus (up to genus 50) is doable
- you need to use a computer algebra system such as Magma or GAP

Further questions - representation

- except for one nice sculpture by Ferguson there are few pictures of a 3D representation of a genus 2 or greater tiling or "soccer ball"
- especially find some nice pictures of “Platonic” Cayley graphs in which there is only one regular polygon
- at least one such tiling arises as the configuration space of a physical system
- find a “nice set” of tiles or polygons which represents the surface after identification of sides